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1989 J. Phys. A: Math. Gen. 22 3597

(http://iopscience.iop.org/0305-4470/22/17/023)

J. Phys. A: Math. Gen. **22** (1989) 3597-3604. Printed in the UK

Transforms of the Coulomb Green function by the form factors of the Graz potential

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Received 8 March 1988, in final form 29 March 1989

Abstract. The term by term separability of the Sturmian function representation of the Coulomb Green function is exploited to construct expressions for the single and double transforms of the corresponding outgoing wave Green function by the form factors of the Graz separable potential.

1. Introduction

In the recent past the Graz group [1] obtained a realistic fit to the N-N interaction in terms of a separable potential. For a particular partial wave characterised by l the form factor of this potential in the representation space (r-space) is given by

$$g_{x_l,l}(r) = \langle r|g_{x_l,l}\rangle = 2^{-l}(l!)^{-1}r^l \exp(-x_l r) \tag{1}$$

where x_l is the inverse range parameter. It has been found that the integral transforms

$$I(r, \beta_l) = \int_0^\infty \mathrm{d}r'\, G_{\mathrm{cl}}^{(+)}(r, r')g_{\beta_l,l}(r') \tag{2}$$

and

$$I(\alpha_l, \beta_l) = \int_0^\infty \int_0^\infty \mathrm{d}r\, \mathrm{d}r'\, g_{\alpha_l,l}(r)G_{\mathrm{cl}}^{(+)}(r, r')g_{\beta_l,l}(r') \tag{3}$$

of the outgoing wave Coulomb Green function $G_{\mathrm{cl}}^{(+)}(r, r')$ written as [2]

$$G_{\mathrm{cl}}^{(+)}(r, r') = (2\mathrm{i}k)^{2l+1}(rr')^{l+1} \exp[\mathrm{i}k(r+r')] \times \frac{\Gamma(l+1+\mathrm{i}\eta)}{\Gamma(2l+2)} \Phi(l+1+\mathrm{i}\eta, 2l+2; -2\mathrm{i}kr_<)\Psi(l+1+\mathrm{i}\eta, 2l+2; -2\mathrm{i}kr_>) \tag{4}$$

with energy $E = k^2 > 0$ play a crucial role in the studies of on- and off-shell properties of the Coulomb-distorted Graz potential [3]. The quantities Φ and Ψ are the regular and irregular confluent hypergeometric functions and $r_<$ and $r_>$ have their usual meaning. In writing (4) we have suppressed the energy dependence of $G_{\mathrm{cl}}^{(+)}(r, r')$. We shall follow this convention throughout.

The object of this work is to derive an uncomplicated method for evaluating the integrals in (2) and (3). To that end we shall deal with the Sturm series representation [4] of the bound-state Coulomb Green function $G_{\mathrm{cl}}(r, r')$, evaluate its single transform with the form factor in (1) and then analytically continue the result to obtain $I(r, \beta_l)$ in terms of Gaussian hypergeometric functions which are absolutely convergent on the entire unit circle. Further, we shall see that the result for $I(\alpha_l, \beta_l)$ can be obtained directly from

$$I(\alpha_l, \beta_l) = \int_0^\infty \mathrm{d}r\, g_{\alpha_l, l}(r) I(r, \beta_l). \tag{5}$$

The Green operator corresponding to $G_{\mathrm{cl}}(r, r')$ is given by [4]

$$G_{\mathrm{cl}} = \sum_{n=l+1}^{\infty} -\frac{n}{n - s/\kappa} G_{0l}|\lambda_n l\rangle\langle\lambda_n l|G_{0l}. \tag{6}$$

Here the Sturm states $|\lambda_n l\rangle$ represent the 'interaction strength' eigenfunctions and are defined as the eigenstates of $V_l G_{0l}$ [4]:

$$V_l G_{0l}(-\kappa^2)|\lambda_n l\rangle = \lambda_n|\lambda_n l\rangle \qquad n = l+1, l+2, \ldots \tag{7}$$

with the fixed energy $-\kappa^2 < 0$. The quantities V_l and G_{0l} stand for the partially projected potential and free-particle Green operator. The states $|\lambda_n l\rangle$ satisfy the orthonormality relation

$$\langle\lambda_{n'} l|G_{0l}|\lambda_n l\rangle = -\delta_{n'n}. \tag{8}$$

When the operator V_l corresponds to the Coulomb potential, $V_{\mathrm{c}}(r) = -2S/r$, the eigenvalues λ_n are given by

$$\lambda_n = S/n\kappa_n. \tag{9}$$

There exists a simple relation between the Coulomb bound states $|K_n l\rangle$ and Sturmian states $|\lambda_n l\rangle$, and we have

$$|K_n l\rangle = 2^{1/2}\kappa_n G_{0l}|\lambda_n l\rangle. \tag{10}$$

The Sturm states form a complete set with the closure relation

$$-G_{0l}^{-1} = \sum_{n=l+1}^{\infty} |\lambda_n l\rangle\langle\lambda_n l|. \tag{11}$$

In § 2 we evaluate the integrals in (2) and (3). We present some concluding remarks in § 3.

2. Results for $I(r, \beta_l)$ and $I(\alpha_l, \beta_l)$

From (4) we see that certain indefinite integrals are implied in (2) and (3) when $G_{\mathrm{cl}}^{(+)}(r, r')$ is written in terms of energy eigenfunctions. Buchholz [5] has considered these indefinite integrals in some detail. Johnson and Hirschfelder [6] observe that use of the results of Buchholz and their concomitant reduction do not reduce (2) and (3) in compact analytical form.

In contrast to (4), equation (6) provides us with a term-by-term separable representation for the Coulomb Green function. This will therefore help us circumvent the characteristic difficulties associated with the above noted indefinite integrals. The price we pay for this is that we must now deal with the series arising from (6) and be very careful about its convergence at positive energies. The single transform in (2) can be related to the mixed representation $\langle r|G_{cl}|g_{\beta_l,l}\rangle$ of the Green operator in (6) as follows.

From (6) we write

$$\langle r|G_{cl}|g_{\beta_l,l}\rangle = \sum_{n=l+1}^{\infty} -\frac{n}{n-S/\kappa_n}\langle r|G_{0l}|\lambda_n l\rangle\langle\lambda_n l|G_{0l}|g_{\beta_l,l}\rangle. \tag{12}$$

Both factors $\langle r|G_{0l}|\lambda_n l\rangle$ and $\langle\lambda_n l|G_{0l}|g_{\beta_l,l}\rangle$ in (12) can be evaluated in a rather straightforward way. For example, in view of (10), $\langle r|G_{0l}|\lambda_n l\rangle$ is simply related to the Coulomb bound-state energy eigenfunction written as [4]

$$\langle\kappa_n l|r\rangle = \left(\frac{\kappa_n}{n}\right)^{1/2}\left(\frac{\Gamma(n-l)}{\Gamma(n+l+1)}\right)^{1/2}(2\kappa_n r)^{l+1}\exp(-\kappa_n r)L_{n-l-1}^{2l+1}(2\kappa_n r). \tag{13}$$

The associated Laguerre polynomial $L_p^{\alpha}(Z)$ in (13) is related to the ${}_1F_1(\ )$ (confluent hypergeometric) function by

$$L_p^{\alpha}(Z) = \frac{\Gamma(p+\alpha+1)}{\Gamma(\alpha+1)\Gamma(p+1)}\,{}_1F_1(-p, \alpha+1; Z). \tag{14}$$

The factor $\langle\lambda_n l|G_{0l}|g_{\beta_l,l}\rangle$ is found in the form

$$\langle\lambda_n l|G_{0l}|g_{\beta_l,l}\rangle = (l!)^{-1}\kappa_n^l\left(\frac{2\kappa_n}{n}\right)^{1/2}(\beta_l^2-\kappa_n^2)^{-l-1}\left(\frac{\Gamma(n+l+1)}{\Gamma(n-l)}\right)^{1/2}Z^n \tag{15}$$

where

$$Z = (\beta_l-\kappa_n)(\beta_l+\kappa_n)^{-1}. \tag{16}$$

In writing (15) we have used the result of the standard integral $\int_0^{\infty}\exp(-xt)t^{a}L_p^{a}(t)\,dt$ given by Magnus *et al* [7]. Equations (10) and (12)–(15) can now be combined to write

$$\langle r|G_{cl}|g_{\beta_l,l}\rangle = -2^{-l}(l!)^{-1}\left(\frac{2\kappa_n}{\beta_l+\kappa_n}\right)^{2l+1}\frac{1}{(\beta_l+\kappa_n)\Gamma(2l+2)}r^{l+1}\exp(-\kappa_n r)$$
$$\times\sum_{m=0}^{\infty}\frac{\Gamma(m+2l+2)Z^m}{(m+l+1-S/\kappa_n)m!}\,{}_1F_1(-m, 2l+2; 2\kappa_n r) \qquad m=n-l-1. \tag{17}$$

Rearranging the terms in the infinite series in (17) we get

$$\langle r|G_{cl}|g_{\beta_l,l}\rangle$$
$$= -2^{-l}(l!)^{-1}(\beta_l+\kappa_n)^{-1}r^{l+1}\exp(-\kappa_n r)$$
$$\times\sum_{m=0}^{\infty}\frac{[-(\beta_l-\kappa_n)r]^m}{(m+l+1-S/\kappa_n)m!}\,{}_2F_1(1, -S/\kappa_n-l; m+l+2-S/\kappa_n; Z). \tag{18}$$

While summarising the recent progress in the study of Coulomb Green functions and propagators Blinder [8] has pointed out that, in addition to a denumerable set of Sturm functions, there exists a corresponding continuum set. Both sets are complete over the same domain as that of the discrete hydrogenic wavefunctions augmented by a continuum set. The continuum Sturmians are obtained from the denumerable ones by means of an analytic continuation similar to that used for the hydrogenic wavefunctions. The Green functions for energies $E<0$ and $E>0$ are also analogously expressed. The expression in (18) involving ${}_2F_1(\ )$ functions may therefore be analytically continued from real positive K into complex κ plane. Hence, replacing κ_n by $-ik$ and $-S/\kappa_n$ by $\mathrm{i}\eta$ we obtain from (18)

$$I(r,\beta_l)=-2^{-l}(l!)^{-1}(\beta_l-\mathrm{i}k)^{-1}r^{l+1}\exp(\mathrm{i}kr)$$
$$\times\sum_{m=0}^{\infty}\frac{[-(\beta_l+\mathrm{i}k)r]^m}{(m+l+1+\mathrm{i}\eta)m!}\,{}_2F_1\left(1,\mathrm{i}\eta-l;\,m+l+2+\mathrm{i}\eta;\,\frac{\beta_l+\mathrm{i}k}{\beta_l-\mathrm{i}k}\right). \qquad (19)$$

It is of interest to note that, for all m, the ${}_2F_1(\ )$ functions in (19) are absolutely convergent.

From (5) and (19) the double transform $I(\alpha_l,\beta_l)$ is obtained as

$$I(\alpha_l,\beta_l)=2^{-2l}(l!)^{-2}(\alpha_l-\mathrm{i}k)^{-1}(\beta_l-\mathrm{i}k)^{-1}$$
$$\times\left(\frac{2\mathrm{i}k}{(\alpha_l-\mathrm{i}k)(\beta_l-\mathrm{i}k)}\right)^{2l+1}\sum_{m=0}^{\infty}\frac{\Gamma(m+2l+2)}{(m+l+1+\mathrm{i}\eta)m!}\left(\frac{2\mathrm{i}k(\beta_l+\mathrm{i}k)}{(\alpha_l-\mathrm{i}k)(\beta_l-\mathrm{i}k)}\right)^m$$
$$\times\,{}_2F_1\left(m+2l+2,\,m+l+1+\mathrm{i}\eta;\,m+l+2+\mathrm{i}\eta;\,\frac{\beta_l+\mathrm{i}k}{\beta_l-\mathrm{i}k}\right). \qquad (20)$$

We now replace the ${}_2F_1(\ )$ function by its integral representation and perform the summation involved to get

$$I(\alpha_l,\beta_l)=-2^{-2l}(l!)^{-2}\frac{\Gamma(2l+2)(\alpha_l+\beta_l)^{-2l-1}}{(l+1+\mathrm{i}\eta)(\alpha_l-\mathrm{i}k)(\beta_l-\mathrm{i}k)}$$
$$\times\,{}_2F_1\left(1,\mathrm{i}\eta-l;\,l+2+\mathrm{i}\eta;\,\frac{(\alpha_l+\mathrm{i}k)(\beta_l+\mathrm{i}k)}{(\alpha_l-\mathrm{i}k)(\beta_l-\mathrm{i}k)}\right). \qquad (21)$$

The result in (21) is in maximal reduced form. This could also be obtained by a differential equation method recently derived by two of us [9]. In the appendix we present a useful check on our results in (19) and (21). We rederive them by using tabulated integrals only.

3. Conclusion

In this paper we have found the single and double transforms of $G_{\mathrm{cl}}^{(+)}(r,r')$ by the form factors of a rank-one separable potential. This potential falls short in producing the ${}^1S_0$ effective range parameters and fitting the P- and D-wave data. The Graz group has parametrised a new separable rank-two potential (Graz II) [10] which meets the demand of a precise description of the p-p interaction for $l\leqslant 2$. Thus, evaluation of the transforms of $G_{\mathrm{cl}}^{(+)}(r,r')$ by the form factors of the Graz II potential is of considerable physical interest. We have verified that this problem does not involve any new mathematical complication and, in fact, all results can be expressed in terms of (19) and (21) and their derivatives with respect to the inverse range parameter.

Acknowledgments

This work was supported in part by the Department of Atomic Energy, Government of India. One of the authors (UL) is grateful to the Council of Scientific and Industrial Research, Government of India.

Appendix. Direct evaluation of the results in (19) and (21)

To evaluate $I(r, \beta_l)$ we begin by introducing the regular Coulomb Green function written as

$$\begin{aligned} G_{\mathrm{cl}}^{(R)}(r, r') &= \frac{1}{(2l+1)}(2\mathrm{i}k)^{2l+1}(rr')^{l+1} \\ &\quad \times \exp[\mathrm{i}k(r+r')][\bar{\Phi}(l+1+\mathrm{i}\eta, 2l+2; -2\mathrm{i}kr)\Phi(l+1+\mathrm{i}\eta, 2l+2; -2\mathrm{i}kr') \\ &\quad - \Phi(l+1+\mathrm{i}\eta, 2l+2; -2\mathrm{i}kr)\bar{\Phi}(l+1+\mathrm{i}\eta, 2l+2; -2\mathrm{i}kr')] \qquad r'<r \\ &= 0 \qquad r'>r \end{aligned} \tag{A1}$$

where

$$\bar{\Phi}(a, c; Z) = Z^{1-c}\Phi(a-c+1, 2-c; Z). \tag{A2}$$

From (1), (2), (4) and (A1) we have

$$\begin{aligned} I(r, \beta_l) &= \frac{2^{-l}(l!)^{-1}}{2\mathrm{i}k} r^{l+1} \\ &\quad \times \exp(\mathrm{i}kr)\left[\frac{1}{(2l+1)}\left(\bar{\Phi}(l+1+\mathrm{i}\eta, 2l+2; -2\mathrm{i}kr)\int_0^r \mathrm{d}(-2\mathrm{i}kr')\right.\right. \\ &\quad \times \exp[-(\beta_l+\mathrm{i}k)r'](-2\mathrm{i}kr')^{2l+1}\exp(2\mathrm{i}kr')\Phi(l+1+\mathrm{i}\eta, 2l+2; -2\mathrm{i}kr') \\ &\quad - \Phi(l+1+\mathrm{i}\eta, 2l+2; -2\mathrm{i}kr)\int_0^r \mathrm{d}(-2\mathrm{i}kr') \\ &\quad \left. \times \exp[-(\beta_l+\mathrm{i}k)r'](-2\mathrm{i}kr')^{2l+1}\exp(2\mathrm{i}kr')\bar{\Phi}(l+1+\mathrm{i}\eta, 2l+2; -2\mathrm{i}kr')\right) \\ &\quad + \frac{\Gamma(l+1+\mathrm{i}\eta)}{\Gamma(2l+2)}(2\mathrm{i}k)^{2l+2}\Phi(l+1+\mathrm{i}\eta, 2l+2; -2\mathrm{i}kr)\int_0^\infty \mathrm{d}r' r'^{2l+1} \\ &\quad \left. \times \exp[-(\beta_l-\mathrm{i}k)r']\Psi(l+1+\mathrm{i}\eta, 2l+2; -2\mathrm{i}kr')\right]. \end{aligned} \tag{A3}$$

Expanding $\exp[-(\beta_l+\mathrm{i}k)r']$ in power series and using the standard integral

$$\begin{aligned} &\int_0^\infty \exp(-ax)x^{s-1}\Psi(b, d; \mu x)\,\mathrm{d}x \\ &\quad = \frac{\Gamma(s)\Gamma(1+s-d)}{a^s\Gamma(1+b+s-d)}\,{}_2F_1(b, s; 1+s+b-d; 1-\mu/a) \\ &\quad \mathrm{Re}\, s>0 \qquad 1+\mathrm{Re}\, S>\mathrm{Re}\, d \end{aligned} \tag{A4}$$

we can rewrite (A3) in the form

$$I(r,\beta_l)=\frac{2^{-l}(l!)^{-1}}{2\mathrm{i}k}r^{l+1}$$

$$\times\exp(\mathrm{i}kr)\Bigg[\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\frac{1}{(2l+1)}\Bigg(\bar{\Phi}(l+1+\mathrm{i}\eta,2l+2;-2\mathrm{i}kr)\int_0^r \mathrm{d}(-2\mathrm{i}kr')$$

$$\times\exp(2\mathrm{i}kr')(-2\mathrm{i}kr')^{n+2l+1}\Phi(l+1+\mathrm{i}\eta,2l+2;-2\mathrm{i}kr')$$

$$-\Phi(l+1+\mathrm{i}\eta,2l+2;-2\mathrm{i}kr)\int_0^r \mathrm{d}(-2\mathrm{i}kr')$$

$$\times\exp(2\mathrm{i}kr')(-2\mathrm{i}kr')^{n+2l+1}\bar{\Phi}(l+1+\mathrm{i}\eta,2l+2;-2\mathrm{i}kr')\Bigg)$$

$$+\frac{1}{(l+1+\mathrm{i}\eta)}\left(\frac{2\mathrm{i}k}{\beta_l-\mathrm{i}k}\right)^{2l+2}{}_2F_1\left(l+1+\mathrm{i}\eta,2l+2;l+2+\mathrm{i}\eta;\frac{\beta_l+\mathrm{i}k}{\beta_l-\mathrm{i}k}\right)$$

$$\times\Phi(l+1+\mathrm{i}\eta,2l+2;-2\mathrm{i}kr)\Bigg] \qquad \text{(A5)}$$

where

$$\rho=(\beta_l+\mathrm{i}k)/2\mathrm{i}k. \qquad \text{(A6)}$$

Babister [11] has shown that

$$\frac{1}{(c-1)}\Bigg(\Phi(a,c;Z)\int^Z \mathrm{e}^{-Z'}Z'^{\sigma+c-2}\bar{\Phi}(a,c;Z')\,\mathrm{d}Z'$$

$$-\bar{\Phi}(a,c;Z)\int^Z \mathrm{e}^{-Z'}Z'^{\sigma+c-2}\Phi(a,c;Z')\mathrm{d}Z'\Bigg)$$

$$=\theta_\sigma(a,c;Z)=\frac{Z^\sigma}{\sigma(\sigma+c-1)}{}_2F_2(1,c+a;\sigma+1;\sigma+c;Z). \qquad \text{(A7)}$$

From (A5) and (A7) we have

$$I(r,\beta_l)=-2^{-l}(l!)^{-1}r^{l+1}$$

$$\times\exp(\mathrm{i}kr)\Bigg[\frac{1}{(\beta_l-\mathrm{i}k)(l+1+\mathrm{i}\eta)}{}_2F_1\left(1,\mathrm{i}\eta-l;l+2+\mathrm{i}\eta;\frac{\beta_l+\mathrm{i}k}{\beta_l-\mathrm{i}k}\right)$$

$$\times\Phi(l+1+\mathrm{i}\eta,2l+2;-2\mathrm{i}kr)+\frac{1}{2\mathrm{i}k}$$

$$\times\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\theta_{n+1}(l+1+\mathrm{i}\eta,2l+2;-2\mathrm{i}kr)\Bigg]. \qquad \text{(A8)}$$

In writing (A8) we have made use of [7]

$${}_2F_1(a,b;c;Z)=(1-Z)^{c-a-b}{}_2F_1(c-a,c-b;c;Z). \qquad \text{(A9)}$$

Equation (A8) represents the result for $I(r,\beta_l)$ in (19).

The result for $I(\alpha_l, \beta_l)$ in (21) can be obtained from $\int_0^\infty \mathrm{d}r\, g_{\alpha_l,l}(r) I(r, \beta_l)$ as follows. From (5) and (A8) we obtain

$$I(\alpha_l, \beta_l) = 2^{-2l}(l!)^{-2}\left[-\frac{\Gamma(2l+2)}{(l+1+\mathrm{i}\eta)(\beta_l-\mathrm{i}k)(\alpha_l^2+k^2)^{l+1}}\left(\frac{\alpha_l-\mathrm{i}k}{\alpha_l+\mathrm{i}k}\right)^{\mathrm{i}\eta}\right.$$
$$\times {}_2F_1\left(1, \mathrm{i}\eta-l; l+2+\mathrm{i}\eta; \frac{\beta_l+\mathrm{i}k}{\beta_l-\mathrm{i}k}\right)+\frac{1}{(\alpha_l-\mathrm{i}k)^{2l+3}}\sum_{n=0}^{\infty}\frac{\rho^n\Gamma(2l+2+n)}{(n+1)!}$$
$$\left.\times\left(-\frac{2\mathrm{i}k}{\alpha_l-\mathrm{i}k}\right)^n {}_2F_1\left(1, n+l+2+\mathrm{i}\eta; n+2; \frac{-2\mathrm{i}k}{\alpha_l-\mathrm{i}k}\right)\right]. \qquad \text{(A10)}$$

To derive (A10) we have used [11]

$$\int_0^\infty \mathrm{e}^{-bZ}Z^\nu\theta_\sigma(a, c; pZ)\,\mathrm{d}Z$$
$$=\frac{\Gamma(\nu+\sigma+1)p^\sigma}{\sigma(\sigma+c-1)b^{\nu+\sigma+1}}\,{}_3F_2(1, \nu+\sigma+1, \sigma+a; \sigma+1, \sigma+c; p/b)$$
$$\operatorname{Re}\sigma>0 \qquad \operatorname{Re}(\sigma+c)>1 \qquad \operatorname{Re}\nu>-1 \qquad \operatorname{Re}b>\operatorname{Re}p \qquad \text{(A11)}$$
$$\int_0^\infty \mathrm{e}^{-\lambda Z}Z^\nu\Phi(a, c; pZ)\,\mathrm{d}Z=\frac{\Gamma(\nu+1)}{\lambda^{\nu+1}}\,{}_2F_1(a, \nu+1; c; p/\lambda) \qquad \text{(A12)}$$

and

$${}_3F_2(a, b, c; c, f; Z)={}_2F_1(a, b; f; Z). \qquad \text{(A13)}$$

Unfortunately, (A10) is not in the standard form as obtained in the text. To reduce (A10) to the desired form we transform ${}_2F_1(1, n+l+2+\mathrm{i}\eta; n+2; -2\mathrm{i}k/(\alpha_l-\mathrm{i}k))$ by the relations (A8) and [7]

$${}_2F_1(a, b; c; Z)$$
$$=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1(a, b; a+b-c+1; 1-Z)$$
$$+(1-Z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,{}_2F_1(c-a, c-b; c-a-b+1; 1-Z)$$
$$\text{(A14)}$$

to get

$$I(\alpha_l, \beta_l)=\frac{2^{-2l}(l!)^{-2}}{2\mathrm{i}k(l+1+\mathrm{i}\eta)(\alpha_l-\mathrm{i}k)^{2l+2}}$$
$$\times\sum_{n=0}^{\infty}\frac{\rho^n\Gamma(2l+2+n)}{n!}\,{}_2F_1\left(-n, l+1+\mathrm{i}\eta; l+2+\mathrm{i}\eta; \frac{\alpha_l+\mathrm{i}k}{\alpha_l-\mathrm{i}k}\right). \qquad \text{(A15)}$$

The infinite sum in (A15) can be removed by replacing

$${}_2F_1\left(-n, l+1+\mathrm{i}\eta; l+2+\mathrm{i}\eta; \frac{\alpha_l+\mathrm{i}k}{\alpha_l-\mathrm{i}k}\right)$$

by its integral representation. We thus get

$$I(\alpha_l, \beta_l) = \frac{2^{-2l}(l!)^{-2}\Gamma(2l+2)}{2\mathrm{i}k(\alpha_l - \mathrm{i}k)^{2l+2}}\left(\frac{2\mathrm{i}k}{\beta_l - \mathrm{i}k}\right)^{2l+2} \times \int_0^1 \mathrm{d}t\, t^{l+\mathrm{i}\eta}\left(1 - \frac{(\beta_l + \mathrm{i}k)(\alpha_l + \mathrm{i}k)}{(\beta_l - \mathrm{i}k)(\alpha_l - \mathrm{i}k)}\, t\right)^{-2l-2}. \tag{A16}$$

Interestingly

$$\int_0^1 \mathrm{d}t\, t^{l+\mathrm{i}\eta}\left(1 - \frac{(\beta_l + \mathrm{i}k)(\alpha_l + \mathrm{i}k)}{(\beta_l - \mathrm{i}k)(\alpha_l - \mathrm{i}k)}\, t\right)^{-2l-2} = \frac{1}{(l+1+\mathrm{i}\eta)}\, {}_2F_1\left(2l+2,\ l+1+\mathrm{i}\eta;\ l+2+\mathrm{i}\eta;\ \frac{(\beta_l + \mathrm{i}k)(\alpha_l + \mathrm{i}k)}{(\beta_l - \mathrm{i}k)(\alpha_l - \mathrm{i}k)}\right). \tag{A17}$$

From (A9), (A16) and (A17) we can write

$$I(\alpha_l, \beta_l) = -\frac{2^{-2l}(l!)^{-2}\Gamma(2l+2)(\alpha_l + \beta_l)^{-2l-1}}{(l+1+\mathrm{i}\eta)(\beta_l - \mathrm{i}k)(\alpha_l - \mathrm{i}k)} \times {}_2F_1\left(1, \mathrm{i}\eta - l;\ l+2+\mathrm{i}\eta;\ \frac{(\beta_l + \mathrm{i}k)(\alpha_l + \mathrm{i}k)}{(\beta_l - \mathrm{i}k)(\alpha_l - \mathrm{i}k)}\right) \tag{A18}$$

the standard result in (21).

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